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Multiple scattering of hard γ quanta in a medium

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We consider the problem of multiple scattering of γ quanta with energies of 1–5 MeV in media. An analytical solution of the problem is possible due to the fact that in the mentioned region of the energy, only Compton scattering contributes to the distribution functions of γ quanta and because recoiled electrons may be considered relativistic. We derive asymptotic formulas for the distribution functions of the momenta of γ quanta at distances larger than the γ-quantum mean free path length. We consider both stationary and nonstationary sources of γ quanta.

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I. INTRODUCTION

Calculations of γ-ray penetration in media are required in a variety of applications including medical radiology, industrial inspection, nuclear power plant core and shielding design, and study of different physical phenomena. Although the elementary processes causing the interaction of γ quanta with a medium have been extensively studied both theoretically and experimentally, the complete description of the interactions is exceedingly complicated by multiple scattering. On the other hand, some calculations, for example, the shielding ones, are often greatly simplified by the use of a buildup factor B, which was introduced by White [1] and is defined as a ratio of the total energy flux density to the energy flux density of uncollided photons. A detailed description of modern numerical methods of calculations of the buildup factor was presented in a review article by Harima [2]. However, for other applications information about B alone is not sufficient. For example, atmospheric photochemistry in the presence of an intense source of γ radiation is determined by the strong luminescence (bremsstrahlung) of fast electrons that arise as a result of Compton scattering of γ quanta. A study of this problem consists of three consecutive stages: (i) calculation of the phase-space distribution function $f(t,\mathbf{r},\mathbf{p})$ of the γ quanta, primary and secondary; (ii) calculation of the distribution function of the fast electrons and their bremsstrahlung, and (iii) calculation of the propagation of the radiated light through a region where intense photochemical reactions are taking place. The reactions are determined by the intensity of the luminescence and, in turn, they affect its absorption. Performing these computations requires, as a starting point, knowledge of $f(t,\mathbf{r},\mathbf{p})$.

The following are elementary processes determining the interaction of γ radiation with a medium: (i) the atomic photoeffect, (ii) coherent scattering γ quanta by atoms, (iii) Compton (incoherent) scattering γ quanta by atomic electrons, (iv) pair production in the fields of the nucleus and of the atomic electrons, and (v) photonuclear scattering. A detailed database on these processes is presented in [3]. As one can see in the database, for all chemical elements in a target, there is a region of quanta energies where the Compton effect cross section is much more than cross sections of all the other elementary processes. It is situated about the region 1–5 MeV. That is precisely the region that we consider in the present work. So we can neglect all the elementary processes in comparison with the Compton effect. Besides, at 1–5 MeV we can assume the free and at rest target electron model for each scattering interaction, neglecting the kinetic energy of electrons as well as the influence of the incoherent scattering function $S(\mathbf{q},Z)$, which takes into account that the target electrons are bound in atoms. It is necessary to account for lower energies of quanta [4]. In the present work we derive the asymptotic distribution function $f(t,\mathbf{r},\mathbf{p})$ far from stationary and nonstationary sources of nonpolarized γ quanta. Because of the small angles of scattering of the γ quanta, we can neglect polarization effects. An interesting discussion of polarization effects arising when photons undergo photoeffect, coherent, and Compton scattering by atomic electrons, is presented in [5]. We apply an asymptotic method that uses the Fourier transformation and analytical continuation of the Fourier transformation into a complex plane. Calculations include a contour integration in the vicinity of singularities [6–8]. Although the calculations are rather complicated, the results are fairly simple. They are expressed by Eq. (75) for the stationary case and by Eqs. (99)–(101) for the nonstationary case.

The organization of the paper is as follows. In Sec. II we study the kinetic equation for the distribution function and obtain its expansion in terms of the number of Compton collisions. In Secs. III–VI we present details of the calculation. In Sec. VII we derive the final result for the stationary case. In Sec. VIII we solve the nonstationary problem. In Sec. IX we discuss the results.

II. KINETIC EQUATION AND EXPANSION IN TERMS OF THE NUMBERS OF COMPTON COLLISIONS OF γ QUANTA

Let us examine the kinetic equation for the density of γ quanta $f(t,\mathbf{r},\mathbf{p})$ in the phase space $(\mathbf{r},\mathbf{p})$ corresponding

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to a point, monochromatic, and isotropic source of γ quanta situated at the origin:

$$\frac{\partial f}{\partial t}+\frac{c}{p}(\mathbf{p}\cdot\nabla)f=(\dot{N}/4\pi p^2)\delta(\mathbf{r})\delta(p-P)+S \tag{1}$$

where c is the velocity of light, P and N are the momentum and the output rate of generated γ quanta, and S is the collision integral. The latter consists of the input and output rate components [9]

$$S=S_{\text{in}}-S_{\text{out}} . \tag{2}$$

As it is well known,

$$S_{\text{out}}=c\kappa(p)f , \tag{3}$$

where

$$\kappa(p)=[l_0(p)]^{-1}=\sigma_{\text{tot}}(p)N_e , \tag{4}$$

l_0 and N_e are the mean free path and the density of electrons in the medium, and σ_{tot} is the total cross section of the photon-electron interaction. The value S_{in} is defined by the integral

$$S_{\text{in}}(\mathbf{p})=cN_e\int d\mathbf{p}'f(t,\mathbf{p}',\mathbf{r})[d\sigma_C(\mathbf{p}',\mathbf{p})/d\mathbf{p}] , \tag{5}$$

where $\mathbf{p}'$ and $\mathbf{p}$ are the momenta of initial and scattered γ quanta at the Compton recoil, $d\sigma_C/d\mathbf{p}$ is the differential cross section for the Compton recoil of the γ quantum [10]

$$d\sigma_C(\mathbf{p}',\mathbf{p})=M(\mathbf{p}',\mathbf{p})\delta\left[m_e c+p'-\frac{\epsilon}{c}-p\right]d\mathbf{p} , \tag{6}$$

$$M(\mathbf{p}',\mathbf{p})=m(p',p)=(m_e c^2 r_0^2/2\epsilon p p')U(p',p) , \tag{7}$$

$$U(p',p)=\left[\frac{p'}{p}+\frac{p}{p'}\right]+m_e^2c^2\left[\frac{1}{p}-\frac{1}{p'}\right]^2 -2m_e c\left[\frac{1}{p}-\frac{1}{p'}\right] , \tag{8}$$

ϵ is the energy of the recoiling electron, and m_e and r_0 are the mass and the classical radius of the electron.

We consider the stationary case when $\dot{N}$ and f in Eq. (1) do not depend on t, so that $\partial f/\partial t=0$. For that case, let us consider the Fourier transformation of the function f

$$F(\mathbf{k},\mathbf{p})=\int e^{i\mathbf{k}\cdot\mathbf{r}}f(\mathbf{r},\mathbf{p})d\mathbf{r} . \tag{9}$$

Equations (1)–(9) yield, for the function $F(\mathbf{k},\mathbf{p})$,

$$\begin{aligned}(\kappa-ip^{-1}\mathbf{k}_p)F(\mathbf{k},\mathbf{p}) &=(N_e r_0^2 m_e c^2/2\epsilon p)\\ &\quad\times F(\mathbf{k},\mathbf{p}')U(p',p)\delta\left[m_e c+p'-p-\frac{\epsilon}{c}\right]p'^{-1}d\mathbf{p}'\\ &\quad+(\dot{N}/4\pi p^2 c)\delta(p-P) .\end{aligned} \tag{10}$$

Hereafter, it is convenient to adopt the system of units in which

$$c=1 , \quad m_e=1 , \quad 2\pi N_e r_0^2=1 , \tag{11}$$

so that the values $\kappa,\mathbf{k},\mathbf{p},\mathbf{r},l_0$ are dimensionless. However, in our final results we revert to the primary system. Next, we represent the solution of Eq. (10) as

$$F(\mathbf{k},\mathbf{p})=\sum_{n=0}^{\infty}F^{(n)}(\mathbf{k},\mathbf{p}) , \quad f(\mathbf{r},\mathbf{p})=\sum_{n=0}^{\infty}f^{(n)}(\mathbf{r},\mathbf{p}) , \tag{12}$$

where

$$\begin{aligned}F^{0}(\mathbf{k},\mathbf{p})&=[\dot{N}/4\pi(\kappa p-i\mathbf{k}\cdot\mathbf{p})p]\delta(p-P)\\ &=(\dot{N}/4\pi)\int_0^{\infty}\exp[-(\kappa p-i\mathbf{k}\cdot\mathbf{p})\xi]\\ &\quad\times\delta(p-P)d\xi/p ,\end{aligned} \tag{13}$$

$$\begin{aligned}F^{(n)}(\mathbf{k},\mathbf{p})&=[1/4\pi(\kappa p-i\mathbf{k}\cdot\mathbf{p})\epsilon]\\ &\quad\times\int F^{(n-1)}(\mathbf{k},\mathbf{p}')U(p',p)\\ &\quad\times\delta(1+p'-\epsilon-p)d\mathbf{p}'/p' .\end{aligned} \tag{14}$$

The expansion can be viewed as an expansion in the numbers of the Compton collisions of γ quanta: Eq. (13) is the contribution of the nonscattered γ quanta and the terms in Eq. (14) are the contributions from the quanta that have been scattered n times. Applying (14) n times and taking into consideration Eq. (13), we obtain the result (detailed calculations are presented in the Appendix)

$$\begin{aligned}F^{(n)}(\mathbf{k},\mathbf{p})&=\frac{\dot{N}p^2}{(4\pi)^{n+1}P^2}\\ &\quad\times\int\prod_{m=1}^{n}[U(p_{m-1},p_m)]\delta(\mathbf{p}-\mathbf{p}_n)\\ &\quad\times\prod_{m=0}^{n}\{\exp[-(\kappa_m\rho_m-i\mathbf{k}\cdot\boldsymbol{\rho}_m)]\\ &\quad\times d\boldsymbol{\rho}_m/\rho_m^2\}\end{aligned} \tag{15}$$

provided the direction of the momenta $\mathbf{p}_m$ coincides with that of the vector $\boldsymbol{\rho}_m$, where $\mathbf{p}_m$ is the momentum of the γ quantum after the mth scattering. We recall that the directions of the vectors $\mathbf{p}_m$ (or $\boldsymbol{\rho}_m$) fix all the values p_m (at the definite value $p_0=P$). In Eq. (15) we use the notation $\kappa_m\equiv\kappa(p_m)$.

The inverse Fourier transformation of Eq. (15) yields

$$f^{(n)}(\mathbf{r},\mathbf{p})=(4\pi)^{-n-1}\dot{N}p^2P^{-2}I , \tag{16a}$$

$$I=\int\prod_{m=1}^{n}[U(p_{m-1},p_m)]\delta(\mathbf{p}-\mathbf{p}_n)\delta\left[\mathbf{r}-\sum_{m=0}^{n}\boldsymbol{\rho}_m\right] \times\prod_{m=0}^{n}[\exp(-\kappa_m\rho_m)d\boldsymbol{\rho}_m/\rho_m^2] . \tag{16b}$$

For $n=0$, Eq. (16) yields the expression for the Fourier transform of the function Eq. (13)

$$f^{(0)}(\mathbf{r},\mathbf{p})=\dot{N}(4\pi r^2)^{-1}e^{-\kappa(p)r}\delta\left[\mathbf{p}-P\frac{\mathbf{r}}{r}\right] . \tag{17}$$

In light of this solution, the result expressed by Eq. (16) is almost as obvious as the consequence of n successive

Compton recoils of an initial γ quantum radiated by the point source, so that the vectors $\boldsymbol{\rho}_m$ are the mean free path of the quantum between the mth and $(m+1)$th collisions.

Equations (15) and (16) can be simplified in the case

$$p \gg m_e c^2 \tag{18}$$

or, in terms of (11),

$$p \gg 1 \ . \tag{19}$$

Under this condition, we can approximate the expression for $U(p_{m-1},p_m)$ [Eq. (8)] as

$$U(p_{m-1},p_m)=2\exp(p_{m-1}^{-1}-p_m^{-1}) \tag{20}$$

[to an accuracy of $(p_{m-1}^{-1}-p_m^{-1})^2<p^{-2}$]. Consequently,

$$\prod_{m=1}^{n}[U(p_{m-1},p_m)]=2^n\exp(P^{-1}-p^{-1}) \ . \tag{21}$$

To simplify further calculations, we introduce the distribution functions

$$f_1(\mathbf{r},p)\equiv\int f(\mathbf{r},\mathbf{p})p^2 d\Omega_{\mathbf{p}} \ , \tag{22}$$

$$F_1(\mathbf{k},p)\equiv\int F(\mathbf{k},\mathbf{p})p^2 d\Omega_{\mathbf{p}} \tag{23}$$

instead of the functions $f(\mathbf{r},\mathbf{p}),F(\mathbf{k},\mathbf{p})$; $d\Omega_{\mathbf{p}}$ is an element of a spatial angle of $\mathbf{p}$, while the index 1 reminds us that the functions are one-dimensional distributions. Of course, with the functions (22) and (23), we lose information about the distribution in the direction of the momentum $\mathbf{p}$, but the latter, as we show below, is trivial at $r\to\infty$ under the condition Eq. (19). The functions (22) and (23) can be presented in a form of expansion analogous to Eq. (12). Equations (15), (16), and (21)–(23) yield

$$f_1^{(n)}(\mathbf{r},p)=[\dot{N}p^2P^{-2}/2(2\pi)^{n+1}]\exp(P^{-1}-p^{-1}) \times\int\delta(p-p_n)\delta\left[\mathbf{r}-\sum_{s=0}^{n}\boldsymbol{\rho}_s\right] \times\prod_{m=0}^{n}[\exp(-\kappa_m\rho_m)d\boldsymbol{\rho}_m/\rho_m^2] \ , \tag{24}$$

$$F_1^{(n)}(\mathbf{k},p)=\frac{\dot{N}p^2}{2(2\pi)^{n+1}P^2}\exp\left[\frac{1}{P}-\frac{1}{p}\right] \times\int\delta(p-p_n)\prod_{m=0}^{n}[\exp(-\kappa_m\rho_m+i\mathbf{k}\cdot\boldsymbol{\rho}_m) \times d\boldsymbol{\rho}_m/\rho_m^2] \ . \tag{25}$$

III. CALCULATION OF THE FUNCTIONS $F_1^{(n)}$

To calculate the multidimensional integral (25), let us specify the expression

$$\exp(-\kappa_m\rho_m+i\mathbf{k}\cdot\boldsymbol{\rho}_m)\frac{d\boldsymbol{\rho}_m}{\rho_m^2} =\exp(-\kappa_m\rho_m+i\mathbf{k}\cdot\rho_m z_{\mathbf{k}m})d\rho_m d\Omega_m \ , \tag{26}$$

where $z_{\mathbf{k}m}$ are the cosines of the angles between the vectors $\boldsymbol{\rho}_m$ and $\mathbf{k}$ and $d\Omega_m$ are the spatial angles of the vectors $\boldsymbol{\rho}_m$. Further calculations are simplified considerably if we adopt the approximation

$$z_{\mathbf{k}m}=z_{\mathbf{k}0}\prod_{s=1}^{m}(z_{s,s-1}) \ , \tag{27}$$

where $z_{m,m-1}$ is the cosine of the angle between the vectors $\boldsymbol{\rho}_m$ and $\boldsymbol{\rho}_{m-1}$. This approximation means that we substitute $z_{\mathbf{k}m}$ by its mean value, an approximation that we call an ensemble average. The validity of the approximation and corrections are discussed in Sec. VI.

Because of (19) and the relation

$$z_{m,m-1}=1-(p_m^{-1}-p_m^{-1}) \ , \tag{28}$$

we may adopt the approximation

$$\prod_{s=1}^{m}(z_{s,s-1})=\exp(P^{-1}-p_m^{-1}) \ . \tag{29}$$

Integrating over all $d\rho_m$ on the right-hand side of Eq. (25) and taking into account Eqs. (26)–(29), we may rewrite Eq. (25) in the form

$$F_1^{(n)}=\frac{\dot{N}p^2}{2P^2}\exp(P^{-1}-p^{-1}) \times\int\delta(p-p_n) \times\prod_{m=0}^{n}\left[\kappa_m-ikz_{\mathbf{k},0}\exp\left[\frac{1}{P}-\frac{1}{p}\right]\right]^{-1} \times dz_{\mathbf{k},0}\prod_{m=1}^{n}(dz_{m,m-1}) \ . \tag{30}$$

Let us change variables of integration and write

$$x_m=p_m^{-1} \tag{31}$$

instead of $z_{m,m-1}$ (28). Using the notation

$$X=P^{-1} \ , \quad x=p^{-1} \ , \tag{32}$$

we rewrite Eq. (30) as

$$F_1^{(n)}(\mathbf{k},p)=\frac{\dot{N}}{2P^2}\exp(X-x)\int_{-1}^{1}\frac{dz_{\mathbf{k},0}}{[\kappa(X)-ikz_{\mathbf{k},0}][\kappa(x)-ikz_{\mathbf{k},0}\exp(X-x)]}$$
$$\times\int_X^x\frac{dx_{n-1}}{[\kappa(x_{n-1})-ikz_{\mathbf{k},0}\exp(X-x_{n-1})]}$$
$$\times\prod_{m=n-2}^{m=1}\left\{\int_X^{x_{m+1}}\frac{dx_m}{[\kappa(x_m)-ikz_{\mathbf{k},0}\exp(X-x_m)]}\right\}$$
$$=(\dot{N}/2P^2)\exp(X-x)\int_{-1}^{1}dz_{\mathbf{k},0}\{[\kappa(X)-ikz_{\mathbf{k},0}][\kappa(x)-ikz_{\mathbf{k},0}\exp(X-x)]\}^{-1}$$
$$\times[(n-1)!]^{-1}\left\{\int_X^x[\kappa(y)-ikz_{\mathbf{k},0}\exp(X-y)]^{-1}dy\right\}^{n-1}. \quad (33)$$

As we show below, the asymptotic behavior of the function $f_1(\mathbf{r},p)$ at $r\to\infty$ is determined by a singularity of the function $F_1(\mathbf{k},p)$ in the complex k plane at $k=i\kappa(X)$ (because of isotropy, the functional f_1, F_1 depends only on the moduluses of the vectors $\mathbf{r},\mathbf{k}$). It is clear that the inner integral in Eq. (33) has a logarithmical divergency at lower limit if $z_{\mathbf{k},0}=-1$ and $k=i\kappa(X)$. To extract the divergent part of this integral we present it in the form

$$-\int_X^x[\kappa(y)-ikz_{\mathbf{k},0}\exp(X-y)]^{-1}dy=[\kappa'(X)+\kappa(X)]^{-1}\ln\{g(X,x,kz_{\mathbf{k},0})[\kappa(X)-ikz_{\mathbf{k},0}]^{-1}\}. \quad (34)$$

It is obvious that $g(X,x,kz_{\mathbf{k},0})$ is an analytical function at $k=i\kappa(X)$ if $z_{\mathbf{k},0}=-1$. Taking into account Eq. (34), we present Eq. (33) in the final form

$$F_1^{(n)}(\mathbf{k},p)=\frac{\dot{N}\exp(X-x)}{(n-1)!2P^2[\kappa'(X)+\kappa(X)]^{n-1}}$$
$$\times\int_{-1}^{1}\{[\kappa(X)-ikz_{\mathbf{k},0}][\kappa(x)-ikz_{\mathbf{k},0}\exp(X-x)]\}^{-1}(\ln\{g(X,x,z_{\mathbf{k},0})/[\kappa(X)-ikz_{\mathbf{k},0}]\})^{n-1}dz_{\mathbf{k},0}. \quad (35)$$

We recall the Eq. (35) is obtained under assumptions (27) and (19).

IV. CALCULATION OF THE SINGULAR PART OF F_1

As follows from Eq. (17), at $p<P$, the functions $f^{(0)}(\mathbf{r},\mathbf{p})$ and $F^{(0)}(\mathbf{k},\mathbf{p})$ do not contribute to the expansion Eq. (12), so we consider instead

$$\tilde{f}_1(\mathbf{r},p)=f_1(\mathbf{r},p)-f_1^{(0)}(\mathbf{r},p)=\sum_{n=1}^{\infty}f_1^{(n)}(\mathbf{r},p), \quad (36)$$

$$\tilde{F}_1(\mathbf{k},p)=F_1(\mathbf{k},p)-F_1^{(0)}(\mathbf{k},p)=\sum_{n=1}^{\infty}F_1^{(n)}(\mathbf{k},p), \quad (37)$$

which coincide with the functions $f_1(\mathbf{r},p)$, $F_1(\mathbf{k},p)$ at $p<P$. Let us substitute Eq. (35) into Eq. (37). After summation, we obtain the result

$$\tilde{F}_1(\mathbf{k},p)=[\dot{N}\exp(X-x)/2P^2]M, \quad (38a)$$

$$M\equiv\int_{-1}^{1}\{[\kappa(X)-ikz][\kappa(x)-ikz]\}^{-1}\times\{g(X,x,ikz)/[\kappa(X)-ikz]\}^q dz, \quad (38b)$$

$$q\equiv[\kappa(X)+\kappa'(X)]^{-1}. \quad (38c)$$

As we show below, the asymptotic behavior of the function $f_1(\mathbf{r},p)$ at $r\to\infty$ is determined by the lowest singularity of the function $\tilde{F}_1(\mathbf{k},p)$ in the upper half plane of the complex k plane. From Eq. (4), it is obvious that $\kappa(x)$ is a monotonical increasing function of $x\equiv(1/p)$ in the domain interesting for us. Thus the lowest singularity of the function $\tilde{F}_1(\mathbf{k},p)$ in the upper half plane of the complex k plane is situated at $k=i\kappa(X)$. The singularity originates when we integrate Eq. (38) in the vicinity of the lower limit. The main singular term of the function Eq. (38) at $k=i\kappa(X)$ is

$$\text{sing}\tilde{F}_1(\mathbf{k},p)=\{\dot{N}[\kappa(X)+k'(X)]/2P^2\times\kappa(X)[\kappa(x)-\kappa(X)\exp(X-x)]\}H, \quad (39a)$$

$$H=\exp(X-x)[g_0(X,x)/\kappa(X)+ik]^q, \quad (39b)$$

$$g_0(X,x)\equiv g[X,x,-i\kappa(X)]. \quad (40)$$

From the definition Eq. (34), we obtain

$$g_0(X,x)=[\kappa(X)+\kappa'(X)](x-X)\exp[I(X,x)], \quad (41)$$

where

$$I(X,x)=\int_X^x\{[\kappa(X)+\kappa'(X)]\times[\kappa(y)-\kappa(X)\exp(X-y)]^{-1}-[1/(y-X)]\}dy. \quad (42)$$

V. THE LEADING ASYMPTOTIC TERM OF THE DISTRIBUTION $\tilde{f}_1$

To calculate the distribution $\tilde{f}_1(\mathbf{r},p)$, let us invert the Fourier transformation of f_1 [which has a form analogous to Eq. (9)], taking into account the isotropy of the functions $\tilde{f}_1(\mathbf{r},p)$ and $\tilde{F}_1(\mathbf{k},p)$, i.e.,

$$\tilde{f}_1(\mathbf{r},p)\equiv\tilde{f}_1(r,p)\ ,\quad \tilde{F}_1(\mathbf{k},p)\equiv\tilde{F}_1(k,p)\ . \tag{43}$$

We obtain

$$\tilde{f}_1(r,p)=(2\pi r^2)^{-1}\int_0^\infty \tilde{F}_1(k,p)k\sin(kr)dk$$
$$=-(i/4\pi^2 r)\int_{-\infty}^{\infty}\tilde{F}_1(k,p)\exp(ikr)k\,dk\ . \tag{44}$$

It is worth noting that the last expression in Eq. (44) is valid if the function $F_1(k,p)$ is continued analytically into the negative axis k so that

$$\tilde{F}_1(-k,p)=\tilde{F}_1(k,p)\ . \tag{45}$$

This condition is fulfilled if we realize the analytical continuation of the function $\tilde{F}_1(k,p)$ into the complex plane k as

$$\tilde{F}_1(k,p)=(4\pi/k)\int_0^\infty f_1(r,p)r\sin(kr)dr\ . \tag{46}$$

In Eq. (44) we may lift the contour of integration up to the lowest singularity (branch point) of the function $\tilde{F}_1(k,p)$ in the upper half plane of the complex k plane, which is situated at $k=i\kappa(X)$. If we lift the contour higher, we must go around, without crossing, the cut of the function $\tilde{F}_1(k,p)$, which begins at the branch point $k=i\kappa(X)$ and which we can chose to direct along the positive direction of the imaginary k axis. At $r\to\infty$, the main contribution to the integral is concentrated near $k=i\kappa(X)$ [6–8]. Taking into account the rule mentioned, we substitute the expression of Eq. (39) into Eq. (44) and obtain the result

$$\tilde{f}_1(r,p)=(\dot{N}/4\pi P^2 r^2)\tilde{f}_1^*\ , \tag{47a}$$

$$\tilde{f}_1^*\equiv[\Gamma(1+q)]^{-1}[\kappa(x)\exp(x-X)-\kappa(X)]^{-1}$$
$$\times\exp[-\kappa(X)r][g_0(X,x)r]^q\ , \tag{47b}$$

where q is defined in Eq. (38c). For the derivation we have used the formula

$$\int_C[\kappa(X)+ik]^{-\nu}e^{ikr}k\,dk$$
$$=2\pi i e^{-\kappa(X)r}[\kappa(X)r^{\nu-1}/\Gamma(\nu)]\ . \tag{47c}$$

VI. CORRECTION TO THE APPROXIMATION OF THE ENSEMBLE AVERAGE

Now let us return to the multidimensional integral (25) and consider the inner integral $\int\exp(-i\mathbf{k}\cdot\boldsymbol{\rho}_n)d\Omega_n$, which can be calculated as

$$\int\exp(-i\mathbf{k}\cdot\boldsymbol{\rho}_n)d\Omega_n=2\int_{-1}^{1}dz_{n,n-1}\int_{z_-}^{z^+}dz_{\mathbf{k}n}\exp(-ik\rho_n z_{\mathbf{k}n})[1-z_{\mathbf{k}n}^2-z_{n,n-1}^2-z_{\mathbf{k},n-1}^2+2z_{\mathbf{k}n}z_{n,n-1}z_{\mathbf{k},n-1}]^{1/2}\ , \tag{48}$$

where

$$z^{\pm}=z_{\mathbf{k},n-1}z_{n,n-1}\pm[(1-z_{n,n-1}^2)(1-z_{\mathbf{k},n-1}^2)]^{1/2}\ . \tag{49}$$

The definitions of $z_{\mathbf{k}n}$ and $z_{n,n-1}$ are given after Eqs. (26) and (27). The inner integral in Eq. (48) can be calculated exactly with the result

$$\int\exp(-i\mathbf{k}\cdot\boldsymbol{\rho}_n)d\Omega_n$$
$$=2\pi\int_{-1}^{1}J_0(k\rho_n[(1-z_{n,n-1}^2)(1-z_{\mathbf{k},n-1}^2)]^{1/2})$$
$$\times\exp(-ik\rho_n z_{n,n-1}z_{\mathbf{k},n-1})dz_{n,n-1}\ , \tag{50}$$

where $J_0(y)$ is the Bessel function. If the argument of the Bessel function is much less than unity in the domain that contributes the most to the integral Eq. (25), one can take $J_0(y)\approx 1$ and the result Eq. (50) implies that the ensemble-average approximation Eq. (27) is valid. To obtain corrections to this approximation, one needs to substitute the ensemble-average values of $\rho_n, z_{n,n-1}, z_{\mathbf{k},n-1}$ into the argument of the Bessel function in Eq. (50). After the substitution, the Bessel function in Eq. (50) becomes a constant and may be brought out of the integral. Analogously, in Eq. (25) one can integrate over all $dz_{\mathbf{k}m}$. Then, the total correction to the ensemble-average approximation for the function $F_1^{(n)}(\mathbf{k},p)$ is expressed by the factor

$$Z^{(n)}=\prod_{m=1}^{n}J_0(k\bar{R}_m[(1-\bar{z}_{m,m-1}^2)(1-\bar{z}_{\mathbf{k},m-1}^2)^2]^{1/2})\ , \tag{51}$$

where

$$R_m=z_{m,m-1}^{-1}\sum_{m'=m}^{n}\rho_{m'}\prod_{m''=m}^{m'}z_{m'',m''-1} \tag{52}$$

and $\bar{R}_m, \bar{z}_{m,m-1}, \bar{z}_{\mathbf{k},m-1}$ are the ensemble averages of $R_m, z_{m,m-1}, z_{\mathbf{k},m-1}$. From Eqs. (28) and (32) it follows that

$$\bar{z}_{m,m-1}=1-(\bar{x}_m-\bar{x}_{m-1})\ . \tag{53}$$

In the singular term of $F_1^{(n)}$, as it has been stated above, we have the equalities

$$k=i\kappa(X)\ , \tag{54}$$

$$z_{\mathbf{k},0}=-1\ . \tag{55}$$

In the ensemble-average approximation from Eqs. (25), (27), and (29) we have the formulas

$$\bar{z}_{\mathbf{k},m}=\bar{z}_{\mathbf{k},0}\prod_{m'=1}^{m}\bar{z}_{m',m'-1}=\exp(X-\bar{x}_m)\ , \tag{56}$$

$$\bar{\rho}_m = [\kappa(\bar{x}_m) + \kappa(X)\bar{z}_{k,m}]^{-1} = [\kappa(\bar{x}_m) - \kappa(X)\exp(X - \bar{x}_m)] \ . \quad (57)$$

From Eqs. (51)–(57) it follows that to compute the factor $Z^{(n)}$ it is necessary to compute $\bar{x}_m$. To that end, we appeal to Eq. (33), taking into account the notation of Eqs. (34) and (40) and the fact that the main contribution to the contour integral (44) is generated at $|\kappa(X) - ikz_{k,0}| \sim r^{-1} \to 0$. As a result, we obtain

$$\ln[rg_0(X,\bar{x}_m)] = (m/n)\ln[rg_0(X,x)] \ , \quad m \geq 1 \ . \quad (58)$$

If Eq. (18) is satisfied, then $X < x \ll 1$ and so $I(X,x) \ll 1$, in accordance with Eq. (42). Then, Eqs. (41) and (58) yield

$$\bar{x}_m - X = \{1/[\kappa(X) + \kappa'(X)]r\}\exp(mL/n) \ , \quad (59)$$

where

$$L = \ln\{r[\kappa(X) + \kappa'(X)](x - X)\} \ , \quad (60)$$

Now, from Eqs. (52), (25), (53), and (29) under condition (18), we have

$$\bar{R}_m \approx \sum_{m'=m}^{n} \bar{\rho}_{m'} = \sum_{m'=m}^{n} [\kappa(\bar{x}_{m'}) - \kappa(X)\exp(X - \bar{x}_{m'})]^{-1} \approx \{1 - \exp[-(n - m + 1)L/n]\} \times \{[\kappa(X) + \kappa'(X)](\bar{x}_m - X) \times [1 - \exp(-L/n)]\}^{-1} \ , \quad (61)$$

$$(1 - \bar{z}_{m,m-1}^2)(1 - \bar{z}_{k,m-1}^2) \approx 4(\bar{x}_m - \bar{x}_{m-1})(\bar{x}_{m-1} - X) = 4(\bar{x}_m - X)^2[1 - \exp(-L/n)]\exp(-L/n) \quad \text{at } m \geq 2 \ , \quad (62a)$$

$$(1 - \bar{z}_{1,0}^2)(1 - \bar{z}_{k,0}^2) \approx 0 \ . \quad (62b)$$

From Eqs. (51), (61), and (62), we find $Z^{(n)}$ at $k = i\kappa(X)$:

$$Z^{(n)} = Y^{(n)} K_n^{n-1} \ , \quad (63)$$

where

$$K_n = I_0\{[\exp(L/n) - 1]^{-1/2} 2\kappa(X)/[\kappa(X) + \kappa'(X)]\} \ , \quad (64)$$

$$Y^{(n)} = \prod_{m=2}^{n} (1/K_n) I_0 \left\{ [\exp(L/n) - 1]'^{1/2} \times \left[1 - \exp\left[-\frac{n - m + 1}{n} L \right] \right] \times 2\kappa(X)/[\kappa(X) + \kappa'(X)] \right\} , \quad (65)$$

and $I_0(y)$ is the Bessel function of imaginary argument. We notice that in the product (65) we may calculate only several terms near $m = n$ as the others are of order unity. When summing Eq. (37) over n, we are interested in $Z^{(n)}$ at $|n - \bar{n}| \ll \bar{n}$, as the sum is contributed by terms with $|n - \bar{n}| \sim (\bar{n})^{1/2} \ll \bar{n}$. At that n, we have

$$Z^{(n)} = K^{n-1} Y \ , \quad (66)$$

where we have used the notation

$$Y \equiv Y^{(\bar{n})} \ , \quad K \equiv K_{\bar{n}} \ . \quad (67)$$

Taking into account the correcting factor Eq. (66) and the condition $[\kappa(X) - ikz_{k,0}] \sim r^{-1}$, from Eqs. (35) and (37), we obtain

$$\bar{n} = \frac{KL}{\kappa(X) + \kappa'(X)} \ , \quad (68)$$

where L is given by Eq. (60). From Eqs. (64)–(68) it follows that

$$K = I_0 \left\{ \left[\exp\left[\frac{\kappa(X) + \kappa'(X)}{K} \right] - 1 \right]^{-1/2} \times 2\kappa(X)/[\kappa(X) + \kappa'(X)] \right\} , \quad (69)$$

$$Y = \prod_{m=1}^{n-1} (1/K) I_0 \left\{ 2\kappa(X) \left[1 - \exp\left[-m\frac{\kappa(X) + \kappa'(X)}{K} \right] \right] \times \left[\exp\left[\frac{\kappa(X) + \kappa'(X)}{K} \right] - 1 \right]^{-1/2} \times [\kappa(X) + \kappa'(X)]^{-1} \right\} . \quad (70)$$

In the product Eq. (70) it is sufficient to retain only several terms near $m = 1$ as the others are close to unity. Taking into account the correcting factor Eq. (66), we can correct the results Eqs. (39) and (47) as

$$f_1(\mathbf{r},p) = (Y\dot{N}/4\pi r^2 P^2) f_1^* \ , \quad (71a)$$

$$f_1^* = \left\{ \Gamma\left[1 + \frac{K}{\kappa(X) + \kappa'(X)} \right] \right\}^{-1} \times \kappa(x)\exp(x - X) - \kappa(X)]^{-1} \times [rg_0(X,x)]^{\{K/[\kappa(X) + \kappa'(X)]\}} \exp[-\kappa(X)r] \ . \quad (71b)$$

VII. FINAL RESULT FOR THE DISTRIBUTION FUNCTION f_1 IN AN ARBITRARY SYSTEM OF UNITS

We recall that (71) is valid in the special system of units Eq. (11). To rewrite it in an arbitrary system it is convenient to present the value $\kappa(P)$ [Eq. (4)] in the form

$$\kappa(p) \equiv \kappa(x) = \sigma_{\text{tot}}(p) N_e = 2\pi r_0^2 N_e \gamma(x) \ , \quad (72)$$

where in the arbitrary system, the definition Eq. (32) becomes

$$X = (m_e c/P) \ , \quad x = (m_e c/p) \ . \quad (73)$$

For the Compton collision [10]

$$\gamma(x) \equiv (\sigma_C / 2\pi r_0^2)$$
$$= x \frac{2(2+x)(1+x)^2 - (3+x)}{(2+x)^2} + \frac{1}{2} x [1 - 2x(1+x)] \ln \left[\frac{2+x}{x} \right] . \quad (74)$$

Let us rewrite the result Eq. (71) in the arbitrary system of units, accounting Eq. (41),

$$f_1(\mathbf{r},p) = \{ Y r_0^2 N_e \dot{N} X^2 / 2\Gamma[1 + (K/\tilde{\gamma})] r m_e c^2 \} \times \exp[(X-x) + \tilde{\gamma}^{-1} I(X,x)] \times [\tilde{\gamma}(x-X) 2\pi N_e r_0^2 r]^{[(K/\tilde{\gamma})-1]} \times \exp[-\kappa(X) r] , \quad (75)$$

where

$$\tilde{\gamma} \equiv \gamma(X) + \gamma'(X) .$$

In the arbitrary system of units, the values K, Y have the form of Eqs. (69) and (70) with the substitution $\kappa(x) \rightarrow \gamma(x)$. At $p = P$, Eq. (75) is substituted with the noncollision contribution

$$f_1^{(0)} = (\dot{N} / 4\pi r^2 c) \exp[-\kappa(x) r] \delta(p - P) . \quad (76)$$

Let us consider a numerical example. Let $P = 5$ MeV. Then

$$X \approx 0.1 , \quad \gamma(X) \approx 0.16 , \quad \gamma' \approx 1.09 ,$$
$$K \approx 1.09 , \quad Y \approx 0.95 .$$

Using this result and Eqs. (75) and (76), we calculate the buildup factor B defined as a ratio of the total energy flux to the energy flux of uncollided quanta. The result is

$$B = 1 + 0.6 (r / l_0)^{0.88} .$$

We plot this result in Fig. 1 versus the numerical result of [2] for water.

VIII. THE NONSTATIONARY CASE

Let us study Eq. (1) with $\dot{N}$ dependent on time and find its propagator. Then

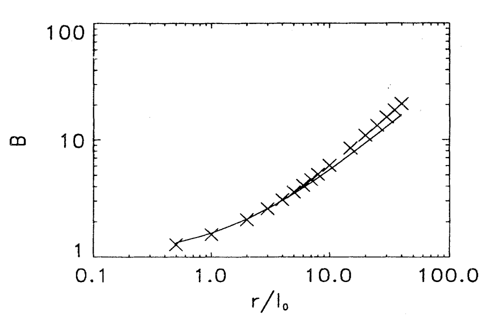

FIG. 1. Buildup factor B at $P = 5$ MeV calculated by using Eqs. (75) and (76) (solid line) versus the numerical result of [2] for water (crosses).

$$f(t,\mathbf{r},\mathbf{p}) = \int_{-\infty}^{t} \dot{N}(\tau) g(t-\tau,\mathbf{r},\mathbf{p}) d\tau , \quad (77)$$

where the propagator $g(t,\mathbf{r},\mathbf{p})$ is the solution of the equation [in the system of units of Eq. (11)]

$$\frac{\partial g}{\partial t} + \frac{\mathbf{p}}{p} \cdot \nabla g + \kappa(p) g = (4\pi p^2)^{-1} \delta(t) \delta(\mathbf{r}) \delta(p - P) + S_{\text{in}}(\mathbf{p}) \quad (78)$$

and S_{in} is analogous to the expression Eq. (5) with the substitution of f with g. If in Eq. (78) we Fourier transform the function g,

$$G(\omega,\mathbf{r},\mathbf{p}) = \int_{-\infty}^{\infty} g(t,\mathbf{r},\mathbf{p}) e^{-i\omega t} dt , \quad (79)$$

the function $G(\omega,\mathbf{r},\mathbf{p})$ satisfies the equation

$$[i\omega + \kappa(p)] G + \frac{\mathbf{p}}{p} \cdot \nabla G = (1/4\pi p^2) \delta(\mathbf{r}) \delta(p - P) + S_{\text{in}} , \quad (80)$$

which coincides with the stationary equation (1) with the substitution of G, κ, and 1 with f, $\kappa + i\omega$, and $\dot{N}$, respectively. Through these substitutions, we can obtain $G_1(\omega,\mathbf{r},p)$ [which is defined quite analogously to the function $f_1(\mathbf{r},p)$] from the result of Eq. (75). The result is

$$G_1(\omega,\mathbf{r},p) = (Y X^2 / 4\pi r) G_1^* , \quad (81a)$$

$$G_1^* \equiv \{\Gamma[1 + K/(\tilde{\kappa} + i\omega)]\}^{-1} \times [(\tilde{\kappa} + i\omega)(x - X) r]^{[K/(\tilde{\kappa}+i\omega)-1]} \times \exp\{X - x - [\kappa(X) + i\omega] r\} , \quad (81b)$$

$$\tilde{\kappa} \equiv \kappa(X) + \kappa'(X) . \quad (82)$$

Now we are interested in the inverse Fourier transform

$$g_1(t,\mathbf{r},p) = (1/2\pi) \int_{-\infty}^{\infty} G_1(\omega,\mathbf{r},p) e^{i\omega t} d\omega , \quad (83)$$

where the functions $G_1(\omega,\mathbf{r},p), g_1(t,\mathbf{r},p)$ are defined in analogy to Eqs. (22) and (23). Substituting Eq. (81) into Eq. (83), the result can be presented as

$$g_1(t,\mathbf{r},p) = \phi_1(t - r,\mathbf{r},p) , \quad (84)$$

where

$$\phi_1(\tau,\mathbf{r},p) = (8\pi^2 r^2)^{-1} A , \quad (85a)$$

$$A \equiv \exp[X - x - \kappa(X) r] \times \int_{\rightarrow} \left\{ \Gamma \left[1 + \frac{K}{\tilde{\kappa} + i\omega} \right] \right\}^{-1} \times [(\tilde{\kappa} + i\omega)(x - X)]^{[K/(\tilde{\kappa}+i\omega)-1]} \times \exp[l(\omega)] d\omega , \quad (85b)$$

$$l(\omega) = i\omega\tau + K \ln(r) / (\tilde{\kappa} + i\omega) . \quad (86)$$

The arrow in the Eq. (85) means that the integral is calculated along the real axes in the complex ω plane. As the integrand in Eq. (85) has no singularities in the lower ω half plane, we can lower the contour of integrating. If

$\tau<0$, the integrand decreases exponentially and we obtain

$$\phi_1(\tau,\mathbf{r},p)=0 \quad \text{at } \tau<0 . \tag{87}$$

To perform the integration in Eq. (85) when $\tau>0$, we can use the saddle-point method as we have the large parameter $\ln(r)$ in the expression for $l(\omega)$ Eq. (86). The necessary condition for the saddle point $l'(\omega_0)=0$ yields $\omega_0=i\tilde{\kappa}\mp i[K\ln(r)/\tau]^{1/2}$. If we take the lower sign in this expression, the saddle point turns out to be located below the singularity of the integrand at $\omega=i\tilde{\kappa}$. We adopt

$$\omega_0(\tau)=i\tilde{\kappa}-i[K\ln(r)/\tau]^{1/2} . \tag{88}$$

From Eqs. (84)–(86) and (88), we obtain the following values of the parameters of the saddle-point method:

$$l_0(\tau)\equiv l(\omega_0(\tau))=-\tilde{\kappa}\tau+2[\tau K\ln(r)]^{1/2} , \tag{89}$$

$$l''(\omega_0)=-2\left[\frac{\tau^3}{K\ln(r)}\right]^{1/2} . \tag{90}$$

Performing the standard saddle-point calculation of the integral Eq. (85) and taking into account the values of the parameters Eqs. (88)–(90), we obtain the result

$$\phi_1(\tau,\mathbf{r},p)=(YX^2/8\pi^{3/2}r^2)\exp[X-x-\kappa(X)r](\Gamma\{1+K/[\tilde{\kappa}+i\omega_0(\tau)]\})^{-1} \times\{[\tilde{\kappa}+i\omega_0(\tau)](x-X)\}^{\{K/[\tilde{\kappa}+i\omega_0(\tau)]-1\}}[K\ln(r)/\tau^3]^{1/4}\exp[l_0(\tau)] . \tag{91}$$

The result is valid if the following conditions are fulfilled:

$$|l^{(n)}(\omega_0)||l''(\omega_0)|^{n/2}\ll 1 \quad \text{for } n>2 , \tag{92}$$

where $l^{(n)}(\omega)$ is the nth derivative of the function $l(\omega)$. From Eqs. (86), (88), and (92), one can obtain the condition

$$[\tau K\ln(r)]^{(2-n)}\ll 1 ,$$

which yields the following domain of validity of the result Eq. (91):

$$\tau\gg[K\ln(r)]^{-1} . \tag{93}$$

As the exponent $l_0(\tau)$ in Eq. (91) contains the large parameter $\ln(r)$ [see Eq. (89)], we may approximate this result by the Gaussian function. With that end in view, we find the position τ_0 of maximum of the function $l_0(\tau)$ Eq. (89),

$$\tau_0=(K/\tilde{\kappa}^2)\ln(r) . \tag{94}$$

From Eqs. (88), (89), and (94), we further obtain

$$\omega_0(\tau_0)=0 , \tag{95}$$

$$l_0(\tau_0)=(K/\tilde{\kappa})\ln(r) , \tag{96}$$

$$\ddot{l}_0(\tau_0)=-\tfrac{1}{2}[\tilde{\kappa}^3/K\ln(r)] . \tag{97}$$

Using these results, we obtain the Gaussian approximation of the function Eq. (91), that is

$$\phi_1(\tau,\mathbf{r},p)=(YX^2/8\pi^{3/2}r)[\Gamma(1+K/\tilde{\kappa})]^{-1} \times[\tilde{\kappa}r(x-X)]^{(K/\tilde{\kappa}-1}[K\ln(r)/\tilde{\kappa}^3]^{-1/2} \times\exp\{X-x-\kappa(X)r -[\tilde{\kappa}^3/4K\ln(r)](\tau-\tau_0)^2\} . \tag{98}$$

Examination of the applicability of the Gaussian approximation to the function of Eq. (91) yields a positive result. Indeed,

$$|l^{(n)}(\tau_0)||\ddot{l}(\tau_0)|^{-1/2n}\sim[\ln(r)]^{1-(1/2)n}\ll 1 \quad \text{for } n>2 .$$

The result (98) has been obtained in the system of units of Eq. (11). To move to an arbitrary system of units, let us present the results of Eqs. (77), (84), and (98) in the form

$$f_1(t,\mathbf{r},p)=\int_0^t\dot{N}(t')\eta[t-t'-r/c-(K/2\pi N_e r_0^2 c\tilde{\gamma}^2) \times\ln(2\pi N_e r_0^2 r)]dt' \tag{99}$$

and

$$\eta(t)=(YX^2/8\pi^{3/2}rm_ec^3)\eta^* , \tag{100a}$$

$$\eta^*\equiv[\tilde{\gamma}(x-X)2\pi N_e r_0^2 r]^{[K/\tilde{\gamma}-1]}[\Gamma(1+K/\tilde{\gamma})]^{-1} \times[\tilde{\gamma}^3/K\ln(2\pi N_e r_0^2 r)]^{1/2}\exp[Q(r,t)] , \tag{100b}$$

$$Q(r,t)=X-x-\kappa(X)r-[\tilde{\gamma}^3/4K\ln(2\pi N_e r_0^2 r)] \times(t^2/4\pi^2N_e^2r_0^4c^2) , \tag{101}$$

where x, X, Y, K, $\tilde{\gamma}$, and $\kappa(X)$ are given by Eqs. (73), (70), (69), (74), (75b), and (4). For $p=P$, instead of Eq. (99) we have

$$f_1^{(0)}(t,\mathbf{r},p)=\frac{\dot{N}(t-r/c)}{4\pi r^2 c}e^{-\kappa(X)r}\delta(p-P) . \tag{102}$$

IX. DISCUSSION

The main result of this paper is the distribution function $f_1(t,\mathbf{r},p)$ in a dense homogeneous medium for γ quanta radiated by a point monochromatic source. The results are presented by either Eq. (75) and/or (76) for the stationary problem or Eqs. (99)–(102) for the nonstationary case. The distribution functions are normalized as

$$\int_0^\infty f_1(t,\mathbf{r},p)dp = n(\mathbf{r},t) , \quad (103)$$

where $n(\mathbf{r},t)$ is the density of the γ quanta.

The distribution function is formed by multiscattering of γ quanta in the medium and the results are valid provided the distances r are larger than the mean free path. Let us recall that the function f_1 is not the complete distribution in the momentum $\mathbf{p}$, but as it follows from the definition Eq. (22), it is only the one-dimensional distribution in p and as such, it contains no information about the angle distribution in the direction of $\mathbf{p}$ (the index 1 of f_1 reminds us of this fact). We have calculated the function f_1 rather than f because, on the one hand, this simplification is instrumental in allowing an analytical treatment and, on the other hand, the angular part of the distribution, as it turns out *a posteriori,* is very simple. Indeed, Eqs. (18) and (28) imply that, at each recoil, the momentum of a γ quantum changes direction only slightly. As recoils are statistically independent, from Eq. (28) one concludes that the mean value of the angle between the vectors $\mathbf{r}$ and $\mathbf{p}$ is

$$\overline{\tfrac{1}{2}\theta^2} \approx 1-\bar{z} = mc(p^{-1}-P^{-1}) \quad (104)$$

(if one uses an arbitrary system of units) and

$$[\overline{(z-\bar{z})^2}]^{1/2} = (mc/\bar{n})(p^{-1}-P^{-1}) , \quad (105)$$

where $\bar{n}$ is the mean value of a number of recoils

$$\bar{n} = (K/\bar{\gamma})\ln[2\pi r_0^2 N_e \gamma r(x-X)] . \quad (106)$$

Because $\bar{n}\to\infty$ as r increases, Eqs. (104) and (105) imply that the momentum of the final γ quanta $\mathbf{p}$ at the point $\mathbf{r}$ is directed in the vicinity of a conic surface, provided the axis of the cone is directed along of the vector $\mathbf{r}$; an angle included between the axis and generatrix is given by Eq. (104).

It is clear that if the source of the γ quanta is not a point source and is not monochromatic, the distribution function can be easily obtained by a proper convolution integral of the function f_1. Our result can be extended also to the case when the medium is nonhomogeneous. In this case, the electron density is a function of $\mathbf{r}$ [i.e., $N_e = N_e(\mathbf{r})$] and one should make the substitution

$$N_e r \to \int_0^r N_e(\mathbf{r}')dr , \quad (107)$$

where the integration is performed along $\mathbf{r}$ as the γ quantum propagates in vicinity of this line. Let us recall once more that all our results are valid in the interval of the energy of the γ quanta (1–5 MeV), where, on the one hand, Eq. (18) is fulfilled and, on the other hand, the scattering γ quanta are contributed mainly by Compton scattering.

APPENDIX

We derive Eq. (15). Applying Eq. (14) n times (starting from $n=1$) and taking into account Eq. (13), we obtain the expression

$$F^{(n)}(\mathbf{k},\mathbf{p}) = \dot{N}(4\pi)^{-(n+1)}p^{-2} \times \int_{\mathbf{p}_0}\cdots\int_{\mathbf{p}_n}(d\mathbf{p}_n)\delta(\mathbf{p}-\mathbf{p}_n)\left[\int_0^\infty \exp\{(i\mathbf{k}\cdot\mathbf{p}_n-\kappa p_n)\xi_n\}d\xi_n\right]\delta(p_0-P) \times \prod_{m=n}^{m=1}(\epsilon_m p_{m-1})^{-1}U(p_{m-1},p_m)\delta(1+p_{m-1}-\epsilon_m-p_m) \times \left[\int_0^\infty \exp\{-(p_{m-1}\kappa_{m-1}-i\mathbf{k}\cdot\mathbf{p}_{m-1})\xi_{m-1}\}d\xi_{m-1}\right](d\mathbf{p}_{m-1}) , \quad (A1)$$

where we have adopted the notation of (17). Let us transform the variables of integration as

$$(\mathbf{p}_m,\xi_m)\to(p_m,\boldsymbol{\rho}_m\equiv\xi_m\mathbf{p}_m) . \quad (A2)$$

As the vector $\mathbf{p}_m$ is parallel to $\boldsymbol{\rho}_m$, the substitution of (A2) into the spherical coordinates is equivalent to the substitution

$$(p_m,\xi_m)\to(p_m,\rho_m\equiv\xi_m p_m) . \quad (A3)$$

Obviously, the Jacobian of the transformation (A3) is equal to p_m. Next, we can integrate all δ functions in Eq. (A1) using the rule

$$\int\delta(1+p_{m-1}-\epsilon_m-p_m)dp_m = \epsilon_m p_m/p_{m-1} , \quad (A4)$$

which can be derived from the conservation equation

$$\mathbf{p}_{m-1} = \mathbf{p}_m + \mathbf{p}_{em} \quad (A5)$$

and the relation

$$\epsilon_m = (1+p_{em}^2)^{1/2} . \quad (A6)$$

In Eqs. (A5) and (A6), $\mathbf{p}_{em}$ is the momentum of the recoiled electron in the mth Compton recoil. After substitution of (A2) and the rule (A4), Eq. (A1) yields Eq. (15).

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